JOURNAL OF APPROXIMATION THEORY 47, 323–325 (1986)

A Relation between Best Approximations in the Chebyshev and the Gauges Senses

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Received March 4, 1985

1. Recently [1] a new method of approximation has been introduced which combines fundamental features of Chebyshev and L^p approximations. In the present note we closely relate Chebyshev polynomials of best approximation to polynomials of best approximation corresponding to this new method.

2. Let $1 \leqslant q < \infty$, $-\infty < a < b < \infty$ and f a real function, continuous on $[a, b]$. Set, [1],

$$\left.\begin{aligned} |||f||| &= \max\left\{\left(\int_\alpha^\beta |f(x)|^q\,dx\right)^{1/q} : a \leqslant \alpha \leqslant \beta \leqslant b, f > 0 \text{ on } (\alpha, \beta), \right. \\ &\qquad \left. \text{or } f < 0 \text{ on } (\alpha, \beta)\right\}, \\ |||f|||_* &= \max\left\{\left(\int_\alpha^\beta |f(x)|^q\,dx\right)^{1/q} : a \leqslant \alpha \leqslant \beta \leqslant b, f \geqslant 0 \text{ on } (\alpha, \beta), \right. \\ &\qquad \left. \text{or } f \leqslant 0 \text{ on } (\alpha, \beta)\right\}. \end{aligned}\right\} \quad (1)$$

(For simplicity we took, in [1], $a = 0$, $b = 1$.) We call $|||f|||$ and $|||f|||_*$ the unstarred and starred gauges of f, respectively. For every integer $n \geqslant 0$, we denote by π_n the set of all real polynomials of degree $\leqslant n$.

3. THEOREM. *Let $-\infty < a < b < \infty$, $n \geqslant 0$ an integer, and f a real function, continuous on $[a, b]$. Suppose $f^{(n+1)}$ exists and never vanishes on (a, b). Let $\hat{p}_{n+1} \in \pi_{n+1}$ be the polynomial of best uniform approximation to $\int_a^x f(y)\,dy$ on $[a, b]$ in π_{n+1}. Take, in (1), $q = 1$. Then $\hat{p}'_{n+1}$ is the unique*

0021-9045/86 $3.00

polynomial in π_n minimizing $|||f-p|||$ there and the unique polynomial in π_n minimizing $|||f-p|||_*$ there. Furthermore,

$$\left.\begin{aligned}\min_{p\in\pi_n} |||f-p||| = |||f-\hat{p}'_{n+1}||| = \min_{p\in\pi_n} |||f-p|||_* &= |||f-\hat{p}'_{n+1}|||_* \\ &= 2\max_{a\leqslant x\leqslant b}\left|\int_a^x f(y)\,dy-\hat{p}_{n+1}(x)\right| \\ &= 2\min_{p\in\pi_{n+1}}\max_{a\leqslant x\leqslant b}\left|\int_a^x f(y)\,dy-p(x)\right|.\end{aligned}\right\} \quad (2)$$

Proof. There are points

$$a\leqslant x_0<x_1<x_2\cdots<x_{n+2}\leqslant b$$

and $\sigma=\pm 1$ such that, for $k=0, 1, 2,..., n+2$, we have

$$\int_a^{x_k} f(x)\,dx-\hat{p}_{n+1}(x_k)=(-1)^k\sigma\max_{a\leqslant x\leqslant b}\left|\int_a^x f(y)\,dy-\hat{p}_{n+1}(x)\right|.$$

Hence, if $0\leqslant k\leqslant n+2$ and $a<x_k<b$, then

$$f(x_k)-\hat{p}'_{n+1}(x_k)=0.$$

By assumption on $f^{(n+1)}$ and by Rolle's theorem, $f-\hat{p}'_{n+1}$ vanishes at at most $n+1$ points of $[a, b]$. Therefore

$$a=x_0<x_1<\cdots<x_{n+2}=b$$

and $f(x)-\hat{p}'_{n+1}(x)\neq 0$ if $a\leqslant x\leqslant b$ and $x\neq x_k$ for $k=1, 2,..., n+1$.

For $k=1, 2,..., n+2$, as

$$\int_{x_{k-1}}^{x_k}[f(x)-\hat{p}'_{n+1}(x)]\,dx=2(-1)^k\sigma\max_{a\leqslant x\leqslant b}\left|\int_a^x f(y)\,dy-\hat{p}_{n+1}(x)\right|,$$

one has

$$(-1)^k\sigma(f-\hat{p}'_{n+1})>0 \qquad \text{on } (x_{k-1}, x_k).$$

By Theorem 3.1 of [1], since, for $k=1, 2,..., n+2$,

$$(-1)^k\sigma\int_{x_{k-1}}^{x_k}[f(x)-\hat{p}'_{n+1}(x)]\,dx=|||f-\hat{p}'_{n+1}|||=|||f-\hat{p}'_{n+1}|||_*,$$

$\hat{p}'_{n+1}$ is the unique polynomial in π_n minimizing $|||f-p|||$ there. By Theorem 3.2 of [1], $\hat{p}'_{n+1}$ is the unique polynomial in π_n minimizing $|||f-p|||_*$ there. Clearly (2).

4. EXAMPLE (cf. [1, Theorem 4.1]). Let $n \geqslant 0$ be an integer and consider the Chebyshev polynomial of the first kind $T_{n+2}(x)$, defined by the identity $T_{n+2}(\cos x) \equiv \cos(n+2)x$. Then $x^{n+2} - 2^{-n-1}T_{n+2}(x) \in \pi_{n+1}$ best uniformly approximates x^{n+2} on $[-1, 1]$ in π_{n+1}. So

$$[x^{n+2} - 2^{-n-1}T_{n+2}(x) - (-1)^{n+2}]/(n+2) \in \pi_{n+1}$$

best uniformly approximates $\int_{-1}^{x} y^{n+1}\, dy$ on $[-1, 1]$ in π_{n+1}. Hence, by the above Theorem, with $a = -1$, $b = 1$,

$$x^{n+1} - 2^{-n-1}(n+2)^{-1}\, T'_{n+2}(x) \equiv x^{n+1} - 2^{-n-1}U_{n+1}(x)$$

is the unique polynomial in π_n minimizing $|||x^{n+1} - p(x)|||$ there and the unique polynomial in π_n minimizing $|||x^{n+1} - p(x)|||_*$ there. Here U_{n+1} is the $(n+1)$th degree Chebyshev polynomial of the second kind. Thus $2^{-n-1}U_{n+1}(x)$ is the unique polynomial minimizing $|||P(x)|||$ among all real polynomials P of degree $n+1$ with leading coefficient 1, and the unique polynomial minimizing $|||P(x)|||_*$ among all such polynomials. Also

$$\begin{aligned} |||2^{-n-1}U_{n+1}||| &= |||2^{-n-1}U_{n+1}|||_* \\ &= 2 \max_{-1 \leqslant x \leqslant 1} |(n+2)^{-1}\, 2^{-n-1}T_{n+2}(x)| = (n+2)^{-1}\, 2^{-n}. \end{aligned}$$

REFERENCE

1. A. PINKUS AND O. SHISHA, Variations on the Chebyshev and L^q theories of best approximation, *J. Approx. Theory* **35** (1982), 148–168.